The Effect of Affiliation on Equilibrium Strategies in *k*-Double Auctions

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Abstract: For *k*-double auctions with $k \in (0,1)$, the effect of a change in the degree of affiliation between buyer and seller values on equilibrium bidding strategies is described. Local changes are shown to depend on how affiliation influences standard inverse hazard rates. While for some equilibria, increased affiliation results in high-value buyers and low-value sellers shading their bids less, for other equilibria increased affiliation can cause either type of trader to shade their bids more. Globally the impact of increased affiliation on bidding strategies is shown to be non-monotonic for a open set of equilibria.

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1. Introduction

Economists have long studied the role of affiliation among buyer values in auction models. In double auction models, the effect of affiliation has received less attention in large part because of the lack of results on the existence of positive-trade equilibria. With a sufficiently large number of buyers and sellers, existence and efficiency results have been developed for double auction models with correlated values by Reny and Perry (2003), Fudenberg, Mobius, and Szeidl (2004), and Cripps and Swinkels (2006). For a very general set of single and double auctions with private and possibly statistically dependent values, Jackson and Swinkels (2005) prove existence of equilibria in which trade occurs with positive probability as long as there exist at least two buyers or at least two sellers.

For the case of one buyer and one seller, Kadan (2005) proves that positive-trade equilibria exist for *k*-double auctions when the buyer and seller values are affiliated and satisfy a condition he refers to as "Bounded Association". Bounded Association together with affiliation implies that trader utilities satisfy the Spence-Mirrlees single crossing property and permits Kadan to characterize the set of strictly increasing equilibria for $k \in (0,1)$.[1] Kadan proves that strictly increasing equilibria of *k*-double auctions with private and affiliated values satisfying Bounded Association are defined by a pair of differential equations. These equations are natural generalizations of the differential equations that Chatterjee and Samuleson (1983) and Satterthwaite and Williams (1989) derived for *k*-double auctions with private and independent values. This similarity allows Kadan to prove the equilibrium correspondence is lower hemi-continuous with respect to an affiliation parameter. It also allows one to investigate the effect of affiliation on equilibrium bidding strategies which is the focus of this paper.

The analysis is based on the following observation. Let $I \subseteq \mathbb{R}$ denote an interval and let $F(c,v)$ denote a joint probability distribution with a non-negative density on $I \times I$ where c denotes the seller's valuation and v denotes the buyer's valuation. According to Kadan, there exists an open subset of affiliated distributions, $\mathscr{F}$, such that for any $k \in (0,1)$ and for any $F \in \mathscr{F}$, the *k*-double auction has a continuum of strictly increasing equilibria. For each k and for each $F \in \mathscr{F}$, the set of strictly increasing equilibria can be indexed by the initial-condition vector $(\hat{c}, \hat{b}, \hat{v})$ where a seller with value $\hat{c}$ bids $\hat{b}$ in equilibrium and a buyer with value $\hat{v}$ bids $\hat{b}$ in equilibrium. Define $C(F,k)$ to be the set of initial

[1] For the cases of k=0 and k=1, neither existence nor the characterization rely on Bounded Association.

conditions associated with the set of strictly increasing equilibria for the k-double auction when the distribution of trader valuations is described by F. Kadan's results are shown to imply that for any k, $k' \in (0,1)$ and for all F, $F' \in \mathscr{F}$, $C(F,k) = C(F',k') \equiv C^*$. For each $(\hat{c},\hat{b},\hat{v}) \in C^*$, I show how the equilibrium through each element of C^* changes as the underlying distribution, F, changes in a manner consistent with increasing affiliation.

Intuitively, as the buyer and seller values become more affiliated, a high-value buyer believes it is more likely she will bargain with a high-value seller. For a given bid less than the buyer's true value, it is more likely that no trade will occur with more affiliation. This will give the buyer an incentive to shade her bid less. At the same time, more affiliation will lead a high-value seller to perceive an increased probability of trade holding her bid constant. This gives the high-value seller an incentive to increase her bid in response to more affiliation. Together these effects of affiliation create an incentive for the traders to adopt steeper strategies: the trader types who are most likely to trade based solely upon their valuations, high-value buyers and low-value sellers, bid closer to their valuations while low-value buyers and high-value sellers submit bids farther from their valuations. Consistent with this intuition, all of Kadan's examples show that increased affiliation makes both buyer and seller strategies steeper. I show that this intuition is incomplete because it requires an increase in affiliation to change the standard inverse hazard rates in a specific direction. While increased affiliation does change the inverse hazard rates in the required direction for some equilibria, it changes the inverse hazard rates in the opposite direction for other equilibria – increased affiliation can make a bidder's strategy flatter. Strategically, equilibrium bidding strategies depend on inverse hazard rates because each trader needs to assess the probability that trade is feasible conditional on her own valuation. Increased affiliation affects the relevant inverse hazard rates in a non-uniform manner, something the above intuition fails to capture.

The effect of increased affiliation is even more complex than would be suggested by a simple comparison of inverse hazard rate formulas. The above comparison of equilibria based on the inverse hazard rates turns out to provide only a local comparison. For a large class of equilibria, increased affiliation causes equilibrium strategies to respond non-monotonically. Local changes in an equilibrium need not persist. That is, an increase in affiliation can create several regions of trader values in which a trader bids more aggressively and several regions in which the same trader bids less aggressively.

Section 2 presents a standard k-double auction model with affiliated trader values and presents Kadan's equilibrium characterization. Section 3 shows how to apply this characterization to identify the range of local differences in equilibria induced by an increase in affiliation. Section 4 investigates the global effect of increased affiliation. The lack of persistence of local differences is shown to be a generic property.   Section 5 offers concluding remarks.

2. The Model

For comparison purposes, the notation in this paper is essentially the same as that used by Kadan. A buyer and a seller will negotiate the trade of an indivisible object using a *k*-double auction. This auction requires each trader to simultaneously submit a bid to a third party. Let *b* denote the buyer's bid and let *s* denote the seller's bid. The object is traded if $b \geq s$ at the price $kb+(1-k)s$ for $k \in [0,1]$. If $b<s$, no trade occurs and no money changes hands.

Each trader knows her value for the object but does not know the other trader's value. Let $c \in I$ denote the seller's value and let $v \in I$ denote the buyer's value. Unless otherwise stated, assume without loss of generality that $I = [0,1]$. The traders have common ex ante beliefs concerning the distribution of values. F(c,v) denotes the joint distribution and f(c,v) denotes the joint density. The seller's interim belief about the buyer's value is described by the conditional distribution, $F(c|v)$ (and $f(c|v)$), while the buyer's interim belief about the seller's value is described by the conditional distribution, $F(v|c)$ (and $f(v|c)$). As is common in private information models, the analysis generates inverse hazard rate expressions. The inverse hazard rates that arise in conjunction with *k*-double auctions are

$$R(c|v) = F(c|v)/f(c|v)$$

and

$$T(v|c) = (F(v|c) - 1)/f(v|c)\,.$$

The following assumptions are used throughout the paper.

A1. For all $c, v \in I$, $f(c,$v) is strictly positive and C^1.

A2. For all $c, v \in I$, $R_c(c|v) > 0$ and $T_v(v|c) > 0$.[2]

A3. Affiliation: For all $c_1,c_2,v_1,v_2 \in I$ such that $c_1 < c_2$ and $v_1 < v_2$, $f(c_1,v_1)f(c_2,v_2) \geq f(c_1,v_2)f(c_2,v_1)$

A4. Bounded Association: For all $c, v \in I$, $v\, f_v(c|v)/f(c|v) > -1$ and $(1-c)\, f_c(v|c)/f(v|c) < 1$.

Assumption A1 ensures that the support of the conditional densities is always equal to the support of the unconditional densities. Assumption 2 is equivalent to assuming that $F(c|v)$ is log-concave in c for all v and that $F(v|c)$ is log-concave in v for all c. This assumption is stronger than the corresponding assumption in Kadan (2005). The stronger assumption simplifies some of the analysis without limiting the range of qualitative changes affiliation has on equilibrium bidding strategies. Assumption A3 restricts

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[2]Subscripts denote partial differentiation.

the type of statistical dependency between the trader values. As Kadan (2005) points out, with just two traders, affiliation is equivalent to assuming f satisfies the Monotone Likelihood Ratio Property. From Milgrom and Weber (1982), we know that affiliation also implies $R_v \leq 0$ and $T_c \leq 0$. Assumption A4 is used by Kadan (2005) to limit the extent to which a change in a trader's value changes her beliefs about the other trader's value. Bounded Association is satisfied if the trader values are statistically independent and if the affiliation in the joint distribution is not too strong. Let $\mathscr{F}$ denote the set of all joint distributions satisfying Assumptions A1-A4.

Definition. *Let $F^0, F^1 \in \mathscr{F}$ have the same marginal distributions. F^1 is more affiliated than F^0 if for all $v \in I$ and for all $c, c' \in I$ such that $c' > c$,*

$$\frac{\partial}{\partial v}\frac{f^1(v|c')}{f^1(v|c)} \geq \frac{\partial}{\partial y}\frac{f^0(v|c')}{f^0(v|c)} \tag{1}$$

and if (1) holds with strict inequality for some $v \in I$.

The Monotone Likelihood Ratio Property (which is equivalent to affiliation with 2 random variables) requires that each derivative in (1) be non-negative. Thus, ranking the derivatives for two distributions is a natural way to define increasing affiliation.[3]

Lemma 1. *Let $F^0, F^1 \in \mathscr{F}$ and assume that F^1 is more affiliated than F^0. Define $\epsilon(c,v)=f^1(c,v)-f^0(c,v)$ and define $f^{\lambda}(c,v)= (1-\lambda)f^0(c,v)+\lambda\epsilon(c,v)$. Let $F^{\lambda}(\cdot,\cdot)$ denote the associated distribution. There exists $\lambda^* > 0$ such that for all $\lambda \in (0,\lambda^*)$, $F^{\lambda} \in \mathscr{F}$.*

Proof. From the continuity properties of f^0 and f^1, f^{λ} must converge pointwise to f^0 as λ goes to zero. Thus, f^{λ} must satisfy A1, A2, and A4 for λ sufficiently close to zero. To show that A3 is also satisfied note $F^0 \in \mathscr{F}$ implies for any $c_1 < c_2$ and for any $v_1 < v_2$ that,

$$f^0(c_1,v_1)f^0(c_2,v_2) - f^0(c_1,v_2)f^0(c_2,v_1) > 0$$

and $F^1 \in \mathscr{F}$ implies

$$f^0(c_1,v_1)f^0(c_2,v_2) - f^0(c_1,v_2)f^0(c_2,v_1) + \Gamma(c_1,c_2,v_1,v_2) > 0$$

where

[3] Alternative inequality definitions of affiliation such as the definition used in A3 or the definition requiring the cross partial of the logarithm of the joint density to be non-negative yield equivalent pointwise definitions of increasing affiliation.

$$\Gamma(c_1,c_2,v_1,v_2) = \epsilon(c_1,v_1)\epsilon(c_2,v_2) - \epsilon(c_1,v_2)\epsilon(c_2,v_1)$$
$$+ f^0(c_1,v_1)\epsilon(c_2,v_2) + f^0(c_2,v_2)\epsilon(c_1,v_1) - f^0(c_1,v_2)\epsilon(c_2,v_1) - f^0(c_2,v_1)\epsilon(c_1,v_2).$$

Since F^1 is more affiliated than F^0, $\Gamma(c_1,c_2,v_1,v_2) > 0$. If we scale $\epsilon(\cdot,\cdot)$ by a positive factor λ, $\Gamma(\cdot,\cdot,\cdot,\cdot)$ will remain strictly positive. Hence, F^{λ} is affiliated for all $\lambda \in [0,1]$. ***Q.E.D.***

Let $S{:}I \rightarrow I$ and $B{:}I \rightarrow I$ denote the seller and buyer strategies. Given a strategy profile (S,B) and a realization of values, c and v, trader payoffs when $B(v) \geq S(c)$ are

$$v - kB(v) - (1-k)S(c)$$

for the buyer and

$$kB(v) + (1-k)S(c) - c$$

for the seller; otherwise each trader's payoff is zero. (S,B) is a Bayesian equilibrium if

$$S(c) \in argmax_s \int (kB(v) + (1-k)s - c)1_{\{s \leq B(v)\}} dF(v|c)$$

and

$$B(v) \in argmax_b \int (v - kb - (1-k)S(c))1_{\{b \geq S(c)\}} dF(c|v).$$

As in Satterthwaite and Williams (1989) and Kadan (2005), I restrict attention to equilibria with the following properties:

B1: $S(\cdot)$ and $B(\cdot)$ are continuous and strictly increasing (and hence differentiable a.e.)

B2: For all $c \in I$, $c \leq S(c) \leq 1$.

B3: For all $v \in I$, $0 \leq B(v) \leq v$.

B4: For $c \geq B(1)$, $S(c) = c$.

B5: For $v \leq S(0)$, $B(v) = v$.

B6: $S(\cdot)$ is C^1 on $[0,B(1)]$.

B7: $B(\cdot)$ is C^1 on $[S[0],1]$.

A strategy profile, and hence an equilibrium, is regular if conditions B1-B7 are satisfied.

The following theorem is due to Kadan (2005).

Theorem 2. *Suppose $k \in (0,1)$ and $F \in \mathscr{F}$.*

(i) There exist a continuum of regular equilibria of the k-double auction.

(ii) (S,B) is a regular equilibrium of the k-double auction if, and only if, for c $\leq$ B(1) and v $\geq$ S(0),

$$\mathrm{k}S'(c)R(c|B^{-1}(S(c))) + S(c) = B^{-1}(S(c)) \tag{2}$$

and

$$(1-k)B'(v)T(v|S^{-1}(B(v))) + B(v) = S^{-1}(B(v)). \tag{3}$$

3. Local Effects of Affiliation

Using the technique developed in Satterthwaite and Williams (1989), define $c(b) = S^{-1}(b)$ and define $v(b) = B^{-1}(b)$. Then, (2) and (3) are equivalent to

$$\dot{c}(b) = kR(c(b)|v(b))/(v(b) - b) \tag{4}$$

and

$$\dot{v}(b) = -(1-k)T(v(b)|c(b))/(b - c(b)). \tag{5}$$

Combined with the tautology $\dot{b} = 1$, (4) and (5) define a vector field. Each regular equilibrium corresponds to the solution defined by this vector field through a point $(\hat{c},\hat{b},\hat{v})$ such that $0 < \hat{c} < \hat{b} < \hat{v} < 1$. At this point, $S(\hat{c}) = B(\hat{v}) = \hat{b}$. Thus, the set of initial conditions that index the set of regular equilibria of any k-double auction with $k\in(0,1)$ is $C^* = \{(\hat{c},\hat{b},\hat{v})|0 < \hat{c} < \hat{b} < \hat{v} < 1\}$. Let $\underline{b} = S(0)$ and let $\bar{b} = B(1)$. Then Assumptions B4 and B5 imply that $\dot{c}(b) = 1$ for all $b > \bar{b}$ and that $\dot{v}(b) = 1$ for all $b < \underline{b}$. Trade occurs with positive probability only at bids between $\underline{b}$ and $\bar{b}$.

Now consider two distributions from $\mathscr{F}$ which I denote by F^0 and F^1. Let (R^0,T^0) and (R^1,T^1) denote the inverse hazard rates for each distribution and let $(c^i(b),v^i(b))$ denote a regular equilibrium for $i\in\{0,1\}$ such that $c^i(\hat{b}) = \hat{c}$ and $v^i(\hat{b}) = \hat{v}$. That is, both equilibria have the same initial condition. Then,

$$\dot{c}^1(b) - \dot{c}^0(b) = \frac{kR^1(c^1(b)|v^1(b))}{v^1(b) - b} - \frac{kR^0(c^0(b)|v^0(b))}{v^0(b) - b} \tag{6}$$

and

$$\dot{v}^1(b) - \dot{v}^0(b) = \frac{(1-k)T^0(v^0(b)|c^0(b))}{b - c^0(b)} - \frac{(1-k)T^1(v^1(b)|c^1(b))}{b - c^1(b)}. \tag{7}$$

Since $c^0(\hat{b}) = c^1(\hat{b}) = \hat{c} < \hat{b}$ and $v^0(\hat{b}) = v^1(\hat{b}) = \hat{v} > \hat{b}$, (6) and (7) imply

$$\dot{c}^1(\hat{b}) - \dot{c}^0(\hat{b}) = k[R^1(\hat{c}|\hat{v}) - R^0(\hat{c}|\hat{v})]/(\hat{v} - \hat{b}) \tag{8}$$

and

$$\dot{v}^1(\hat{b}) - \dot{v}^0(\hat{b}) = (1-k)[T^0(\hat{v}|\hat{c}) - T^1(\hat{v}|\hat{c})]/(\hat{b} - \hat{c}). \tag{9}$$

Theorem 3. *Fix $k\in(0,1)$. Let $F^0,F^1 \in \mathscr{F}$ and let $(c^i(\cdot),v^i(\cdot))$ denote the equilibrium of the k-double auction with distribution F^i through the point $(\hat{c},\hat{b},\hat{v}) \in C^*$ for $i \in \{0,1\}$. Then $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ if, and only if, $\mathrm{R}^1(\hat{c}|\hat{v}) > R^0(\hat{x}|\hat{v})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$ if, and only if, $T^1(\hat{v}|\hat{c}) < T^0(\hat{v}|\hat{c})$.*

In a neighborhood of $\hat{b}$, $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ implies $(c^1(b) - c^0(b))(b - \hat{b}) > 0$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$ implies $(v^1(b) - v^0(b))(b - \hat{b}) > 0$. Thus, Theorem 3 also provides conditions under which a change in the value distribution will lead locally higher value buyers and locally lower value sellers to reduce the difference between their bids and their values while leading locally lower value buyers and locally higher value sellers to do the opposite. Four cases are possible

(I) $\dot{c}^1(\hat{b}) < \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$,

(II) $\dot{c}^1(\hat{b}) < \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$,

(III) $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$, and

(IV) $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$.

Example 1. Farlie-Gumbel-Morgenstern distributions

Let $G_1(c)$ and $G_2(v)$ denote marginal distributions. For any $\alpha\in(-1,1)$, the FGM copula defines the joint distribution

$$F^\alpha(c,v) = G_1(c)G_2(v)(1 + \alpha(1 - G_1(c))(1 - G_2(v))). \tag{10}$$

For any $\alpha \in (-1,1)$, the marginal distributions of $F^\alpha(\cdot,\cdot)$ are $G_1(\cdot)$ and $G_2(\cdot)$ (with densities $g_1(\cdot)$ and $g_2(\cdot)$), $f(c|v) = g_1(c)[1 + \alpha(1-2G_1(c))(1-2G_2(v))]$, and $F(c|v) = G_1(c)[1 + \alpha(1-G_1(c))(1-2G_2(v))]$. The trader values are independently distributed when $\alpha=0$ and they are positively affiliated when $\alpha>0$. If the marginals are uniform, Kadan (2005) proves the joint distribution will satisfy Bounded Association if $\alpha < 1/3$.

Lemma 4. *Let F^α and F^β be FGM distributions with the same marginal distributions. Then F^α is more affiliated than F^β if, and only if, $\alpha > \beta$.*

Proof. Given (10),

$$\frac{\partial^2}{\partial\alpha\partial v}\frac{f^\alpha(v|c')}{f^\alpha(v|c)} = \frac{4(G_1(c') - G_1(c))g_1(c')g2(v)[1-\alpha(1-2G_2(v))(1-2G_1(c))]}{[1+\alpha(1-2G_2(v))(1-G_1(c))]^3 g_1(c)}. \tag{11}$$

The restrictions on α imply that the bracketed terms in both the numerator and the denominator of the

right-hand side of (11) are positive. Hence, the right-hand-side of (11) is positive. ***Q.E.D.***
Lemma 4 allows us to use the parameter α as a measure of affiliation.

Kosmopoulou and Williams (1998) used the FGM distributions with uniform marginals to study the effect of affiliation on the existence of efficient bilateral bargaining mechanisms. Kadan (2005) used the more general family defined in (10) to prove that for the *k*-double auction, strictly increasing equilibrium correspondence is lower hemi-continuous.

Let c_m and v_m denote the median values for $G_1(\cdot)$ and $G_2(\cdot)$. Then by (11),

$$\partial R^{\alpha}(c|v)/\partial\alpha > 0 \text{ if, and only if, } v < v_m \tag{12}$$

and

$$\partial T^{\alpha}(v|c)/\partial\alpha > 0 \text{ if, and only if, } c < c_m. \tag{13}$$

Conditions (12) and (13) imply that an increase in affiliation can generate four qualitatively different changes in equilibrium strategies in a neighborhood about $(\hat{c},\hat{b},\hat{v})$. Let α_0 and α_1 denote values of the affiliation parameter, α, such that $\alpha_1 > \alpha_0$ and let $F^0(c,v)$ and $F^1(c,v)$ denote the associated joint distributions as defined by (10). Furthermore, let $(c^0(\cdot),v^0(\cdot))$ and $(c^1(\cdot),v^1(\cdot))$ denote the regular *k*-double auction equilibria under each distribution that go through the point $(\hat{c},\hat{b},\hat{v})$. If F^{α} is symmetric in *c* and *v*, then $c_m = v_m$ and only cases I-III can arise. Specifically,

(I) $\hat{c} > c_m$ and $\hat{v} > v_m$ implies $\dot{c}^1(\hat{b}) < \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) > \dot{v}^0(\hat{b})$,

(II) $\hat{c} < c_m$ and $\hat{v} > v_m$ implies $\dot{c}^1(\hat{b}) < \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$, and

(III) $\hat{c} < c_m$ and $\hat{v} < v_m$ implies $\dot{c}^1(\hat{b}) > \dot{c}^0(\hat{b})$ and $\dot{v}^1(\hat{b}) < \dot{v}^0(\hat{b})$.

Case II corresponds to the examples in Kadan (2005) that are consistent with the intuition described in the introduction that increased affiliation makes the equilibrium strategies steeper.

In Figures 1a-1d, the top two lines are equilibrium seller strategies and the bottom two lines are buyer strategies for *k*=1/2. Buyer and seller values are distributed over [0,1] with uniform marginal distributions. The solid lines are equilibrium strategies for α=0 and the dashed lines are equilibrium strategies for α=.95. The large value of α was used to create enough separation between the equilibria for visualization purposes only. The qualitative differences between the equilibria seen in the figures exist for any positive value of α less than one. The initial conditions for Figures 1a-1c were chosen to coincide with the linear equilibrium first identified by Chatterjee and Samuelson (1983).

Figure 1a is analogous to Figure 4 in Kadan (2005) and corresponds to Case II. Increased affiliation results in traders who trade with higher probabilities (low-value sellers and high-value buyers) bidding closer to their true values and traders who trade with lower probabilities bidding farther from

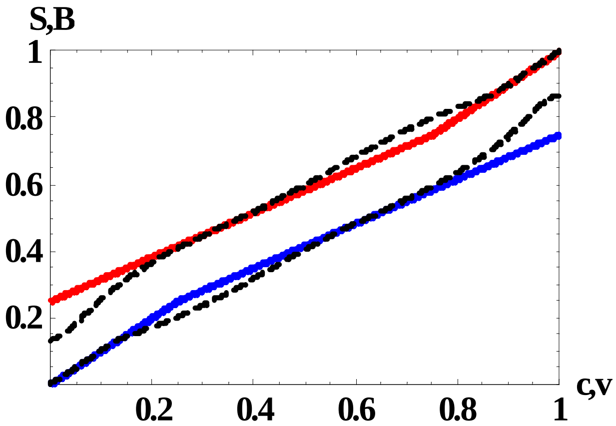

Figure 1a: $(\hat{c},\hat{b},\hat{v})=(3/8,1/2,5/8)$

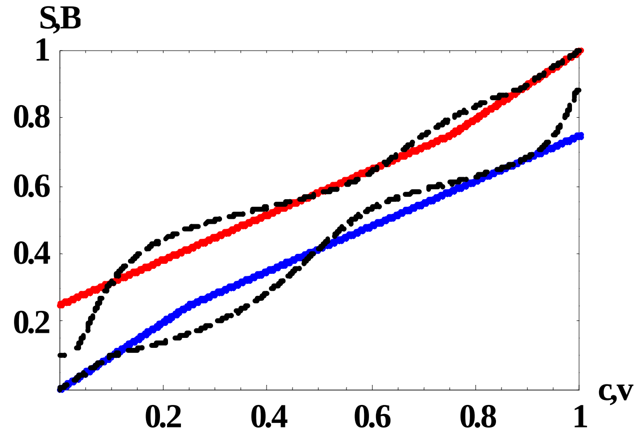

Figure 1b: $(\hat{c},\hat{b},\hat{v})=(5/8,2/3,7/8)$

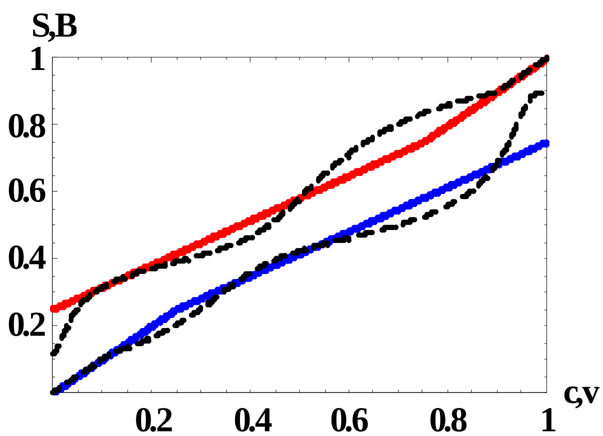

Figure 1c: $(\hat{c},\hat{b},\hat{v})=(1/8,1/3,3/8)$

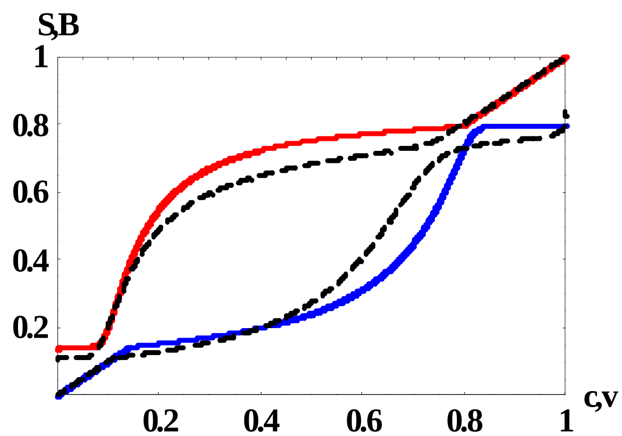

Figure 1d: $(\hat{c},\hat{b},\hat{v})=(1/10,1/5,2/5)$

their true values. Figure 1b corresponds to Case I. Affiliation makes the seller's strategy steeper at c=5/8 but it makes the buyer's strategy flatter at v=7/8. (Although it is hard to see, the buyer's strategy for α=.95 does cut below the strategy line for α=0 at v=7/8.) Thus, for a buyer with a value just below 7/8, increased affiliation causes her to bid closer to her true value and for a buyer with a value just above 7/8, increased affiliation causes her to bid farther from her true value. Figure 1c corresponds to Case III. A seller with a value just above 1/8 will reduce her increment over her value while a seller with a value just below 1/8 will increase her increment. Figure 1d presents another Case III example. In this figure, the baseline independent-type equilibrium is a non-linear equilibrium.

Example 2. Let the joint distribution of trader values be a bivariate normal such that each marginal distribution has mean μ and standard deviation σ and such that the correlation between the valuations is ρ. Kadan (2005) uses this distribution in an example pertaining to k=1. Since his existence theorems for $k\in(0,1)$ require a compact support, truncated bivariate normal distributions would need to be used to calculate equilibria. The following calculations are robust to symmetric truncations. Define

$$\Gamma(c,v,\mu,\sigma,\rho) = (c-\mu-\rho(v-\mu))^2/(2(1-\rho^2)\sigma^2).$$

Then,

$$R^{\mu,\sigma,\rho}(c|v) = \frac{\int_{w=-\infty}^{c} e^{-\Gamma(w,v,\mu,\sigma,\rho)}dw}{e^{-\Gamma(c,v,\mu,\sigma,\rho)}}$$

and

$$\partial R^{\mu,\sigma,\rho}/\partial\rho \propto \int_{w=-\infty}^{c} e^{-\Gamma(w,v,\mu,\sigma,\rho)}[\Gamma_\rho(c,v,\mu,\sigma,\rho) - \Gamma_\rho(w,v,\mu,\sigma,\rho)]dw. \tag{14}$$

Similarly,

$$\partial T^{\mu,\sigma,\rho}/\partial\rho \propto \int_{w=v}^{\infty} e^{-\Gamma(w,c,\mu,\sigma,\rho)}[\Gamma_\rho(w,c,\mu,\sigma,\rho) - \Gamma_\rho(v,c,\mu,\sigma,\rho)]dw. \tag{15}$$

Lemma 2 in Kadan (2005) proves that random variables distributed according to a bivariate normal distribution with $\rho\geq 0$ are affiliated. Moreover, for $v' > v$,

$$\frac{\partial^2}{\partial\rho\partial c}\frac{f(c|v')}{f(c|v)} = \frac{(1+\rho^2)(v'-v)}{(\rho^2-1)^2\sigma^2} > 0.$$

Thus, increases in ρ reflect increasing affiliation.

The signs of (14) and (15) depend on the sign of $\Gamma_{1\rho}$ where

$$\Gamma_{1\rho}(c,v) = \frac{(1+\rho^2)(\mu - v) + 2\rho(c-\mu)}{(\rho^2-1)^2\sigma^2}.$$

When $\hat{c} < \mu < \hat{v}$, $\Gamma_{1\rho}(w,\hat{v})$ is negative for all $w \le \hat{c}$ and $\Gamma_{1\rho}(w,\hat{c})$ is positive for all $w \ge \hat{v}$. Hence, for such initial conditions $\partial R/\partial\rho < 0$ and $\partial T/\partial\rho > 0$ and Case II arises.[4] Case II will also arise when $\hat{c} < \hat{v} < \mu$ and when $\mu < \hat{c} < \hat{v}$ as long as the values of $\hat{c}$ and $\hat{v}$ are not too extreme. Extreme initial conditions will imply Case I or Case III.

4. Global Effects of Affiliation

Figures 1b-1d show that local changes in bidding strategies due to differences in an affiliation parameter need not persist over the entire range of trader values. This lack of persistence in local changes arises because the global analogs of (8) and (9) include terms to reflect changes in the equilibrium strategies away from $\hat{b}$. In this section, I will show that there are open sets of Case I and Case III equilibria for which an increase in affiliation will cause the buyer (seller) strategies to cross at least three times. That is, the local comparisons between buyer and seller strategies will switch at least twice within this set of equilibria.

For any b,

$$\dot{c}^1(b) - \dot{c}^0(b) = \frac{kR^1(c^1|v^1)}{v^1-b} - \frac{kR^1(c^0|v^1)}{v^1-b} + \frac{kR^1(c^0|v^1)}{v^1-b} - \frac{kR^1(c^0|v^0)}{v^0-b} + \frac{kR^1(c^0|v^0)}{v^0-b} - \frac{kR^0(c^0|v^0)}{v^0-b}. \quad (16)$$

For $b = \hat{b}$, the first and second differences disappear leaving only the third difference which is the same as (8). Let Δ_1 denote the first difference in (16), let Δ_2 denote the second difference, and let Δ_3 denote the third difference. Assumption A2 implies Δ_1 will be positive if, and only if, $c^1(b) > c^0(b)$. Assumption A3 implies Δ_2 will be positive if, and only if, $v^1(b) < v^0(b)$. For FGM distributions, Δ_3 will be positive if, and only if, $v^0(b) < v_m$.

Similarly, for any b,

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[4]To satisfy the compact support requirement in Kadan's theorem, suppose trader values are distributed according to a truncated bivariate normal on $[\mu-\delta,\mu+\delta]^2$. This distribution will satisfy A4 as long as $\delta^2 < (1+\rho)\sigma^2/(2\rho)$.

$$\dot{v}^1(b) - \dot{v}^0(b) = \frac{(1-k)T^0(v^0|c^0)}{b-c^0} - \frac{(1-k)T^0(v^1|c^0)}{b-c^0} + \frac{(1-k)T^0(v^1|c^0)}{b-c^0} - \frac{(1-k)T^0(v^1|c^1)}{b-c^1} + \frac{(1-k)T^0(v^1|c^1)}{b-c^1} - \frac{(1-k)T^1(v^1|c^1)}{b-c^1}. \tag{17}$$

Let δ_1, δ_2, and δ_3 denote the three differences on the right-hand side of (17). Assumption A2 implies δ_1 will be positive if, and only if, $v^1(b) < v^0(b)$. Assumption A3 implies δ_2 will be positive if, and only if, $c^1(b) > c^0(b)$. For FGM distributions, δ_3 will be positive if, and only if, $c^0(b) > c_m$. Combined, these conditions imply that Δ_1 and δ_2 will have the same sign as will Δ_2 and δ_1.

Table 1 reports the signs of the Δ_i and δ_i terms near the initial conditions. The second column reports the signs of the local conditions for Cases I-III. Although these cases are defined at $\hat{b}$, our restriction to regular equilibria implies that the signs of Δ_3 and δ_3 will not change in a neighborhood about $\hat{b}$. For example, with an FGM distribution, the signs of Δ_3 and δ_3 will not change until either $c(b) = c_m$ or $v(b) = v_m$. Columns 3 and 4 report the signs of the new effects that arise once $b \neq \hat{b}$. Column 3 does not apply to all values of b greater then $\hat{b}$, but only those for which the signs of $c^1(b) - c^0(b)$ and $v^1(b) - v^0(b)$ are the same as at $b = \hat{b}$. A similar qualification applies to column 4. For each case, the new effects work in the opposite direction of the local effect for at least one of the traders. With Case I, the non-local effects work against the local effects for the buyer when $b > \hat{b}$ and they work against the local effects for the seller when $b < \hat{b}$. It looks as though the non-local and local effects are reinforcing for the seller when $b < \hat{b}$, but this is only true as long as the sign of $c^1(b) - c^0(b)$ and/or $v^1(b) - v^0(b)$ does not change. Similar results exist for the other two cases. Together the information in Table 1 suggests that the effect of increased affiliation on buyer and seller strategies can be more complicated than the common intuition suggests. Moreover, figures 1b-1d provide evidence that the non-local effects can in fact dominate the local effects.

Note that for the FGM example the inverse hazard rates in (4) and (5) are continuously differentiable in α on (-1,1) and for the bivariate normal example, the inverse hazard rates are continuously differentiable in ρ on (-1,1). From Arnold (1973, Sec. 8, Corollary 10), this differentiability of the vector field with respect to a parameter implies that any regular equilibrium will also be continuously differentiable in the parameter. To use this differentiability to describe more global effects of increased affiliation, we need to develop several intermediate results.

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Table 1: Global Affiliation Effects			
Case	**Local Effects**	$b > \hat{b}$	$b < \hat{b}$
I	$\Delta_3 < 0$	$\Delta_1, \Delta_2 < 0$	$\Delta_1, \Delta_2 > 0$
	$\delta_3 > 0$	$\delta_1, \delta_2 < 0$	$\delta_1, \delta_2 > 0$
II	$\Delta_3 < 0$	$\Delta_1 < 0, \Delta_2 > 0$	$\Delta_1 > 0, \Delta_2 < 0$
	$\delta_3 < 0$	$\delta_1 > 0, \delta_2 < 0$	$\delta_1 < 0, \delta_2 > 0$
III	$\Delta_3 > 0$	$\Delta_1, \Delta_2 < 0$	$\Delta_1, \Delta_2 > 0$
	$\delta_3 < 0$	$\delta_1, \delta_2 < 0$	$\delta_1, \delta_2 > 0$

Lemma 5. *For all regular equilibrium of a k-double auction with k∈(0,1), there exists* $\tilde{v} < 1$ *and* $\tilde{c} > 0$ *such that* $\underline{b} \in [0,\tilde{v}]$ *and* $\bar{b} \in [\tilde{c},1]$.

Proof. As in Section 4 of Satterthwaite and Williams (1989), regular equilibria must satisfy $\dot{v}(\underline{b}) \geq 1$ and $\dot{c}(\bar{b}) \geq 1$. Note that $\lim_{v \to 0} T(v|0)/v = -\infty$, $T(1|0) = 0$, $R(0|1) = 0$, and $\lim_{c \to 1} R(c|1)/(1-c) = \infty$. Then since $-T(v|0)/v$ is strictly decreasing in v and $R(c|1)/(1-c)$ is strictly increasing in c, there exists $\tilde{v} < 1$ and $\tilde{c} > 0$ such that $\underline{b} \in [0,\tilde{v}]$ and $\bar{b} \in [\tilde{c},1]$. ***Q.E.D.***

Eqs. (4) and (5) define a one-parameter family of regular equilibria. This is harder to see when one looks at all possible initial conditions of the form $(\hat{c},\hat{b},\hat{v})$ because each regular equilibrium corresponds to a continuum of such conditions. Lemma 5 focuses attention on two possible indices: $\underline{b}$ and $\bar{b}$. The $\underline{b}$-index corresponds to initial conditions of the form $(0,\underline{b},\underline{b})$ and the $\bar{b}$-index corresponds to initial conditions of the form $(\bar{b},\bar{b},1)$.

Lemma 6. *Let (c(·),v(·)) denote a regular equilibrium of a k-double auction with k∈(0,1). For all* $\underline{b} \in (0,\tilde{v})$, *the vector field defined by (4) and (5) at* $\underline{b}$ *must satisfy one of three conditions*:

(a) if $\dot{c}(\underline{b}) \in (0,\infty)$, *then* $\dot{v}(\underline{b}) = 1+k$; *(b)* $\dot{v}(\underline{b}) \neq 1+k$ *and* $\dot{c}(\underline{b}) = 0$, *or (c)* $\dot{v}(\underline{b}) \neq 1+k$ *and* $\dot{c}(\underline{b}) = \infty$.

For all $\bar{b} \in (\tilde{c},1)$, *the vector field defined by (4) and (5) at* $\bar{b}$ *must satisfy one of three conditions:*

(a′) if $\dot{v}(\bar{b}) \in (0,\infty)$, *then* $\dot{c}(\bar{b}) = 2-k$; *(b′)* $\dot{c}(\bar{b}) \neq 2-k$ *and* $\dot{v}(\bar{b}) = 0$; *or (c′)* $\dot{c}(\bar{b}) \neq 2-k$ *and* $\dot{v}(\bar{b}) = \infty$.

Proof. At $\underline{b}$, $c(\underline{b}) = 0$ and $v(\underline{b}) = \underline{b}$. Thus, direct calculation of $\dot{c}(\underline{b})$ using (4) results in 0/0. Applying L'Hopital's rule to (4) yields

$$\lim_{b \to \underline{b}} \dot{c}(b) = \lim_{b \to \underline{b}} k\dot{c}(b)/(\dot{v}(b) - 1). \tag{18}$$

For $\underline{b} \in (0,\tilde{v})$, $\dot{v}(\underline{b})$ is defined by (5). Then if $\dot{c}(\underline{b}) \in (0,\infty)$, (18) implies $\dot{v}(\underline{b}) = 1 + k$. However, if $\dot{v}(\underline{b}) \neq 1 + k$, (18) implies $\dot{c}(\underline{b}) = 0$ or $\dot{c}(\underline{b}) = \infty$. A similar argument produces the results for $\bar{b}$. ***Q.E.D.***

Lemma 6 implies that the set of equilibria can be divided into three groups I refer to as lower endpoint groups. These three lower endpoint groups correspond to equilibria such that $(\underline{b},\dot{c}(\underline{b})) = (\beta,\dot{c}(\beta))$ where $\beta = \dot{v}^{-1}(1+k)$, or such that $(\underline{b},\dot{c}(\underline{b})) = (\underline{b},0)$, or such that $(\underline{b},\dot{c}(\underline{b})) = (\underline{b},\infty)$. Lemma 7 deals with equilibria with endpoints in the first group, $(\underline{b},\dot{c}(\underline{b})) = (\beta,\dot{c}(\beta))$.

Lemma 7. *Let $F^0, F^1 \in \mathscr{F}$ and let (c^i,v^i) be a regular equilibrium under F^i for $i \in \{0,1\}$ satisfying the initial condition $(\hat{c},\hat{b},\hat{v})$. If $\dot{c}^i(\underline{b}^i) \in (0,\infty)$ for $i \in \{0,1\}$, then $\underline{b}^1 < \underline{b}^0$ if, and only if, $T^1(\underline{b}^0|0) > T^0(\underline{b}^0|0)$. If $\dot{v}^i(\bar{b}^i) \in (0,\infty)$ for $i \in \{0,1\}$, then $\bar{b}^1 > \bar{b}^0$ if, and only if, $R^1(\bar{b}^1|1) < R^0(\bar{b}^0|1)$.*

Proof. From Lemma 6, $\dot{v}^1(\underline{b}^1) - \dot{v}^0(\underline{b}^0) = 0$. Then using a decomposition equivalent to (17) implies

$$\dot{v}^1(\underline{b}^1) - \dot{v}^0(\underline{b}^0) = \frac{(1-k)T^0(\underline{b}^0|0)}{\underline{b}^0} - \frac{(1-k)T^1(\underline{b}^0|0)}{\underline{b}^0} + \frac{(1-k)T^1(\underline{b}^0|0)}{\underline{b}^0} - \frac{(1-k)T^1(\underline{b}^1|0)}{\underline{b}^1} = 0. \tag{19}$$

The first difference in (19) is negative by assumption. Thus, the second difference must be positive. Assumption A2 implies $T(v|c)/v$ is increasing in *y* (remember $T < 0$) so $\underline{b}^1$ must be less than $\underline{b}^0$. An analogous argument proves the statement regarding $\bar{b}^1 - \bar{b}^0$. ***Q.E.D.***

The inverse hazard rate conditions in Lemma 7 are satisfied by all FGM distributions if F^1 is more affiliated than F^0.

Given any two distributions, F^0 and F^1, in the interior of $\mathscr{F}$ such that F^1 is more affiliated than F^0, we can use Lemma 1 to construct a family of distributions in which the parameter λ is associated with increasing affiliation and the vector field defined by (4) and (5) is differentiable in λ. (For the FGM family of distributions, $\lambda=\alpha$). Now consider a regular equilibrium under F^0 with the initial condition $(\hat{c},\hat{b},\hat{v})$ for $\underline{b} < \hat{b} < \bar{b}$ and the regular equilibrium under the more affiliated distribution, F^λ, with the same initial condition. Denote the first equilibrium by (c^0,v^0) and denote the second equilibrium by (c^λ,v^λ). Using Arnold (1973, Sec. 8, Corollary 10), an infinitesimally small value of λ will imply that (c^λ,v^λ) will vary from (c^0,v^0) in a differentiable manner. In particular, this means if (c^0,v^0) is associated with a lower endpoint index that is contained within lower endpoint group (a) from Lemma 6, there exists a λ sufficiently close to zero such that $(c^{\lambda},v^{\lambda})$ will have a lower endpoint index in the same group.

Denote the inverse hazard rates associate with F^{λ} by T^{λ} and R^{λ}. If $T^{\lambda}(\cdot|0)$ is increasing in λ and $R^{\lambda}(\cdot|1)$ is decreasing in λ, then Lemma 7 implies that the global effect of affiliation illustrated in Figures 1b and 1c is generic in the sense that the two buyer strategies and the two seller strategies will cross at least three times. For illustration purposes, consider the Case III example of Figure 1c. It is redrawn in Figure 2. The seller strategies intersect at points A, B, and C. Point B indicates where the two equilibrium seller strategies intersect at the initial condition, $(\hat{c},\hat{b},\hat{v}) = (1/8,1/3,3/8)$. At point B, the seller's strategy under affiliation (dashed curve) is flatter than the seller's strategy under independence.

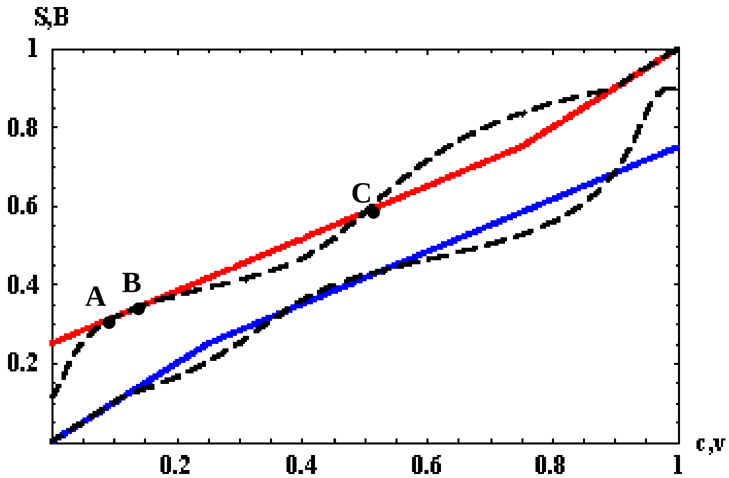

Figure 2: $(\hat{c},\hat{b},\hat{v})=(1/8,1/3,3/8)$

This means a seller with a value just below 1/8 will bid more aggressively when buyer and seller values are more affiliated. (The dashed curve between points A and B lies above the solid curve.) According to Lemma 7, $\underline{b}^1$ must be less than $\underline{b}^0$. Thus, the dashed seller curve cannot stay above the solid seller curve for all seller values below 1/8. The relative slopes at B also imply that a seller with a value just above 1/8 will bid less aggressively as buyer and seller values become more affiliated. But again, according to Lemma 7, $\bar{b}^1$ must be greater than $\bar{b}^0$. Thus, the dashed seller curve cannot stay below the solid seller curve for all seller values greater than 1/8. As this discussion reveals, the next proposition is the result of combining Theorem 3 and Lemma 7.

Proposition 8. *Let F^0, $F^1 \in \mathscr{F}$ and let (c^i,v^i) be a regular equilibrium under F^i for $i \in \{0,1\}$ each satisfying the Case III initial condition $(\hat{c},\hat{b},\hat{v})$. If $\dot{c}^i(\underline{b}^i) \in (0,\infty)$ for $i \in \{0,1\}$ and $T^1(\underline{b}^0|0) > T^0(\underline{b}^0|0)$, then $c^0(\cdot)$ and $c^1(\cdot)$ must be equal for at least three values of b at which trade occurs with positive probability under both equilibria.*

A similar relationship exists between the buyer strategies for the Case I example from Figure 1b. Since there exists a continuum of equilibria with the endpoint condition, $\dot{c}^{\,i}(\underline{b}^i) \in (0,\infty)$, there exists an open set of equilibria of any k-double auction with $k \in (0,1)$ for which the behavior illustrated in Figures 1b and 1c is exhibited. In fact, for equilibria with endpoint conditions $\dot{c}^{\,i}(\underline{b}^i) \in (0,\infty)$ the qualitative local effects of increased affiliation can persist globally only with Case II initial conditions. Finally, the analog to Proposition 8 for the change in buyer strategies in Case III problems and for the change in seller strategies in Case I problems can be derived using the upper endpoint index, $\bar{b}$.

5. Concluding remarks.

The characterization results of Kadan (2005) for regular equilibria of k-double auctions with affiliated random variables allow us to study the effects of increased affiliation on an important class of bargaining mechanisms. Despite the multiplicity of equilibria that exist when $0 < k < 1$, I show how to derive the local effect of an increase in trader valuations on equilibrium strategies. The analysis reveals a wider range of equilibrium responses to increased affiliation than has been suggested by the conventional intuition in the extant literature. I also show that the local effects of increased affiliation need not persist. This lack of persistence results in a non-monotonic change in equilibrium strategies.

Given the variety of equilibrium responses to increased affiliation of trader values with two traders, it is natural to ask how increasing the number of traders affects this variety of responses. Two competing effects will influence the answer to this question. First, as the number of traders increases, the values of $\underline{b}$ and $\bar{b}$ must eventually become more extreme for all equilibria. With $\underline{b}$ sufficiently small or $\bar{b}$ sufficiently large, initial conditions can be chosen near these endpoints to generate Cases I or III. Second, from Rustichini, Satterthwaite, and Williams (1994), adding traders results in vector fields that depend on combinatoric probability expressions that represent a combination of inverse hazard rates. To the extent these combinatorial terms create opposing responses to increased affiliation, the incidence of Cases I and III may decrease with more traders.

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